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APPLICATION OF GEOGEBRA IN TEACHING CLOSURE POINTS IN METRIC SPACES

Umalaxmi Patne

Assistant Professor., Department of Mathematics, VPM's B. N. Bandodkar College of Science (Autonomous), Thane, Maharashtra, India

ABSTRACT

Educational software has revolutionized teaching and research methodologies across various levels of education. Whether it's creating interactive worksheets, developing simulations, conducting data analysis, or facilitating collaborative projects, software tools have become integral to modern teaching and research practices. GeoGebra has emerged as one such significant tool, particularly in the fields of mathematics and science. In this paper, we will see the application of GeoGebra to teach basic concepts of closure points in metric spaces.

KEYWORDS: Geogebra, Metric Space, Open Ball, Closure Point

1. INTRODUCTION

GeoGebra is dynamic mathematics software for teaching and learning mathematics from middle school through college level (Hohenwarter, Markus Preiner, 2007). It combines the ease-of-use of dynamic geometry software with some basic features of computer algebra systems. Although primarily focused on secondary school curriculums, GeoGebra is also an interesting tool for college level courses as it can help to bridge concepts from geometry, algebra, and calculus. GeoGebra was created to help students gain a better understanding of mathematics. You can use it for active and problem-oriented teaching and to foster mathematical experiments and discoveries both in classroom and at home. The software may be used both as a learning and as a teaching tool. On the one hand, students can create constructions from scratch on their own. By doing so, they have the opportunity to solve problems by creating mathematical models and investigating mathematical relations dynamically. On the other hand, the software makes it very easy to create interactive and dynamic online materials for demonstrations or dynamic worksheets (Hohenwarter et al., 2007).

The global community of educators and researchers continually contribute to the development of new worksheets, lesson plans, and methods using GeoGebra, sharing their resources and expertise to support teaching and learning worldwide. These resources can be found at the GeoGebraWiki (www.geogebra.org/wiki, n.d.). This collaborative effort helps enrich educational experiences and promotes innovation in mathematics and science education. Although a wide range of collection is available for topics on Geometry, Calculus and Algebra, it is limited to school level topics. Expanding the use of GeoGebra to college-level topics can greatly enhance the learning experience for students in various disciplines, including mathematics, physics, engineering, and more. One such topic which needs more GeoGebra resources is metric spaces. In this paper, we will see examples of application of GeoGebra in teaching the concept of closure points in metric spaces. GeoGebra applets created for better understanding of closure points in different metric spaces are discussed in this paper. (Hohenwarter & Preiner, 2007).

2. Open Ball

Let (X, d) be a metric space and $p \in X$. An open ball with centre p and radius $r > 0$, denoted by $B(p, r)$, is defined as follows (Kumaresan, 2011):

$$B(p, r) = \{x \in X : d(x, p) < r\}$$

Example 2.1:

Consider $\mathbb{R}^2$ with Euclidean metric d_2 given by $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$. Let $p = (a, b) \in \mathbb{R}^2$ and $r > 0$. Then open ball with centre p and radius r is given by

$$\begin{aligned} B(p, r) &= \{(x, y) \in \mathbb{R}^2 : d_2((x, y), (a, b)) < r\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < r^2\} \end{aligned}$$

GeoGebra applet https://www.geogebra.org/m/b7kx7ye7 can be used for visual demonstration of this open ball (Patne, 2024b).

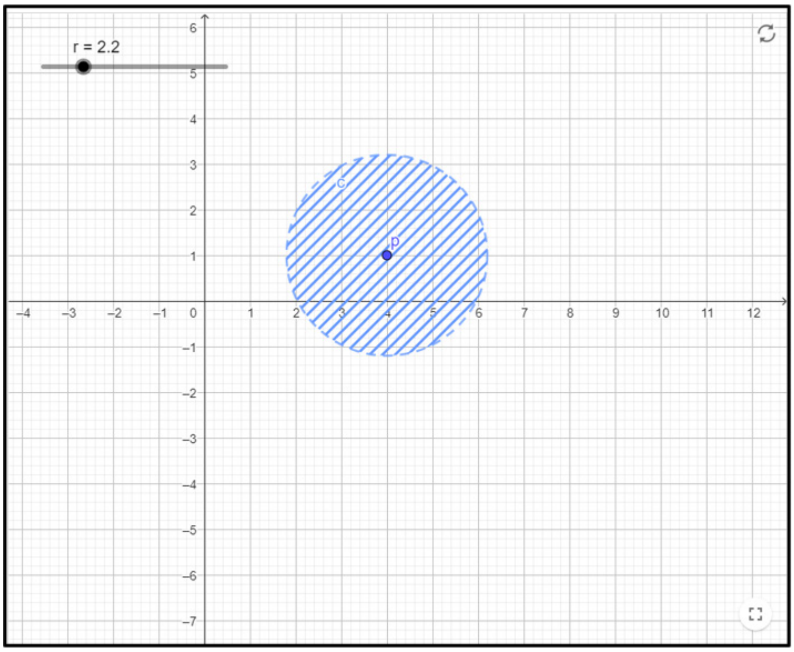

Figure 1: Open Ball in $(\mathbb{R}^2, d_2)$

Example 2.2:

Consider $\mathbb{R}^2$ with sup metric given by $d_\infty(x,y) = \sup\{\,|x_1 - y_1|, |x_2 - y_2|\}$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $p = (a,b) \in \mathbb{R}^2$ and $r > 0$. Then open ball with centre p and radius r is given by

$$\begin{aligned} B(p,r) &= \{(x,y) \in \mathbb{R}^2 : d_\infty\big((x,y),(a,b)\big) < r\} \\ &= \{(x,y) \in \mathbb{R}^2 : \sup\{\,|x-a|, |y-b|\} < r\} \\ &= \{(x,y) \in \mathbb{R}^2 : |x-a| < r \text{ and } |y-b|\} < r\} \end{aligned}$$

GeoGebra applet https://www.geogebra.org/m/xq2kqkth is created for better understanding of the concept of open ball in $(\mathbb{R}^2, d_\infty)$.

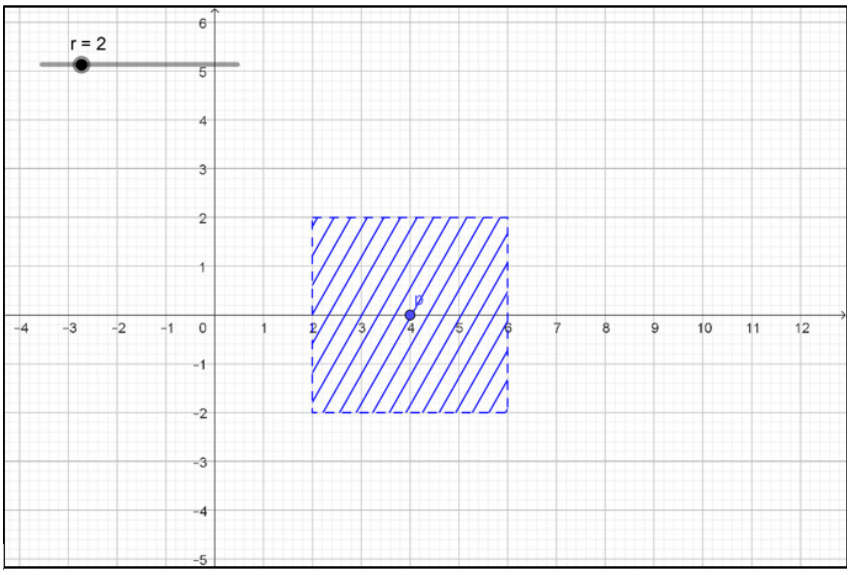

Figure 2:Open Ball in $(\mathbb{R}^2, d_\infty)$

Example 2.3:

Consider the set of all continuous real valued functions defined on interval $[0,1]$ denoted by $C[0,1]$ with sup metric $d(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\}$. GeoGebra applet created for the following example in the metric space $C[0,1]$ is https://www.geogebra.org/m/px5vkjdt (Patne, 2024a)

Let $f(x) = x^3 + 5x - 1$ for $x \in [0,1]$. Then $f \in C[0,1]$. Open ball with centre f will be given by

$$\begin{aligned} B(f,r) &= \{g \in C[0,1] : d(f,g) < r)\} \\ &= \{g \in C[0,1] : \sup\{|f(t) - g(t)| : t \in [0,1]\} < r\} \\ &= \{g \in C[0,1] : |f(t) - g(t)| < r \text{ for all } t \in [0,1]\} \\ &= \{g \in C[0,1] : f(t) - r < g(t) < f(t) + r \text{ for all } t \in [0,1]\} \end{aligned}$$

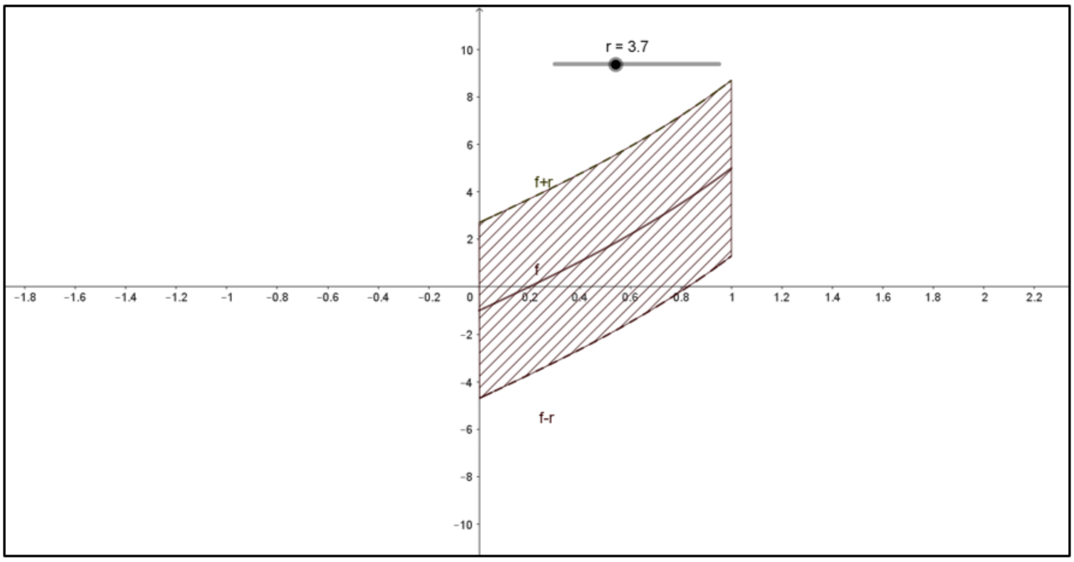

Figure 3: Open Ball in $C[0,1]$ with sup norm

All the above applets are interactive in nature and hence enhance learning, exploration, and understanding by providing hands-on experiences that engage students and facilitate deeper comprehension of concept of open balls in different metric spaces.

3. Closure point

Let *(X,d)* be a metric space and *A* be any non-empty subset of *X*. We say that $p \in X$ is a closure point of *A* if for all *r>0, B(p,r)∩A≠* ∈. Thus *p* is a closure point of *A* if every open ball of *p* intersects with set *A*.

To get a better understanding one needs to visualize open ball and further investigate if it intersects with *A*. Thus interactive applets giving better visualization of this will be highly beneficial to the students. We will see here how to use this interactive applet to teach students the concept of closure points. The applet will help students to check whether the condition of closure point is satisfied or not. The applet will show the given set *A* and provide a slider for radius *r*. A point p will be provided in the applet. Teacher should ask students to change the value of *r* from smallest value to highest value using the slider provided in this applet. Along with changing the value of *r*, students can be encouraged to check if the open ball intersects the given set and hence enabling them to conclude if *p* is a closure point or not. Further students can also move the point p in the same applet and check the condition of closure point again. Hence the applet can also be used to find the closure of a set *A* which is defined as the set of all the closure points of *A* in a metric space *(X,d)*.

Example 3.1:

Consider $\mathbb{R}^2$ with Euclidean metric d_2 given by $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$. Let $A = \{(x,y) \in \mathbb{R}^2 : x > 0\}$. GeoGebra applet created for this set is https://www.geogebra.org/m/e8trq4pr.

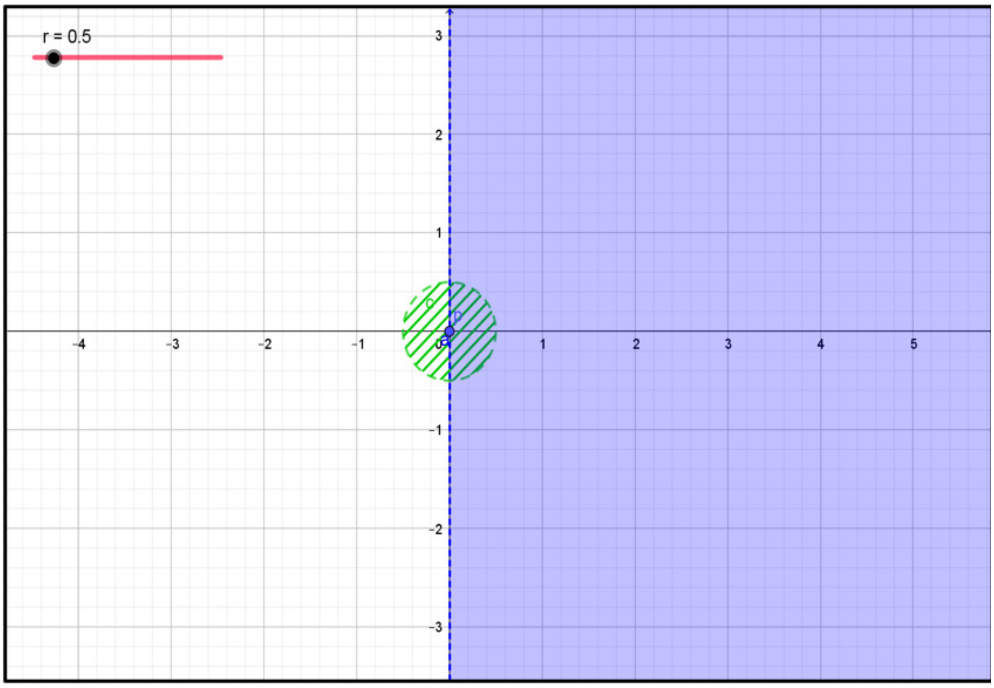

Figure 4:Closure point in $(\mathbb{R}^2, d_2)$ *with* $r = 0.5$

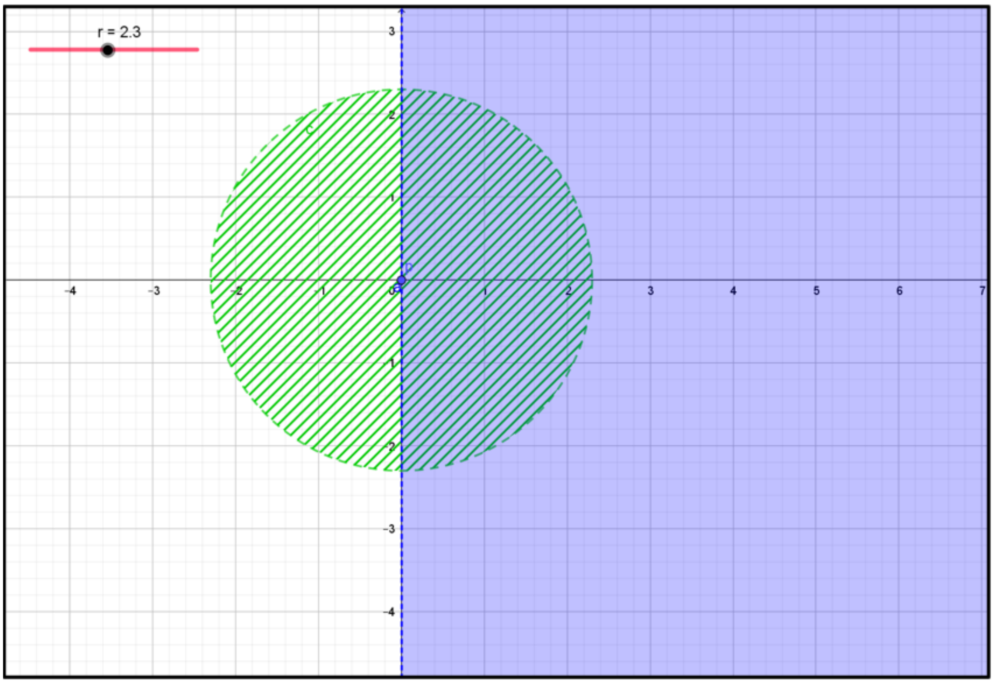

Figure 5:Closure point in $(\mathbb{R}^2, d_2)$ *with* $r = 2.3$

Above figures are snippets of the applet used to check if point *p*=(0,0) is a closure point of set *A*. As seen in above figures, it can be easily concluded that every open ball of p intersects set A and hence is a closure point of *A*.

In the same applet, students are asked to select another point *p*=(0,-1) and change the radius using slider. They were very easily able to conclude that the open ball for $r \leq 1$ did not intersect the set A as shown in the figure below. Hence *p*=(0,- 1) is not a closure point of set *A*.

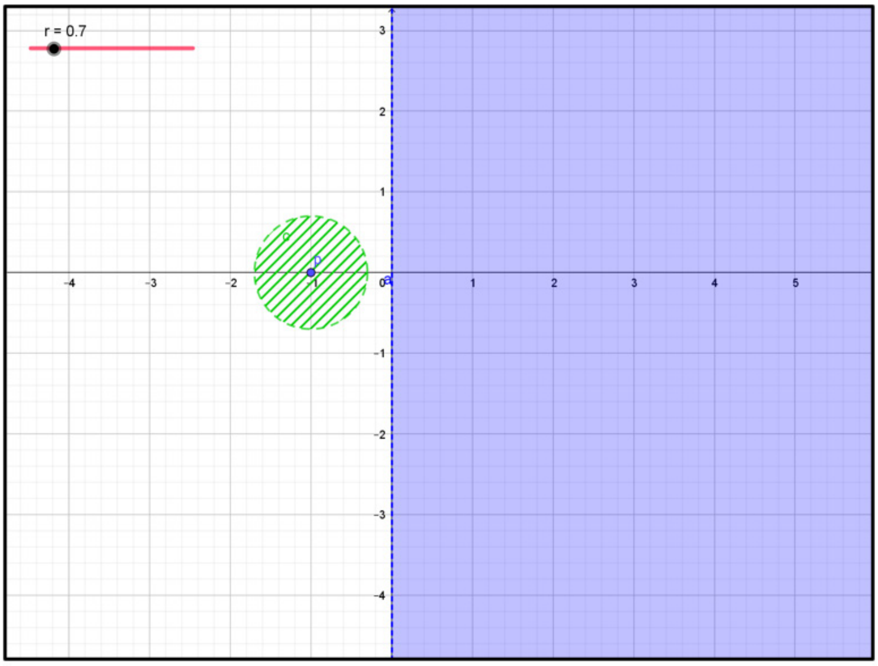

Figure 6: Example of a non-closure point

Example 3.2:

Consider $\mathbb{R}^2$ with sup metric d_∞. Let $G = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. GeoGebra applet created for this set can be found at https://www.geogebra.org/m/tchv8beb. Here students should be asked to consider two cases: $p \in G$ and $p \notin G$.

Case 1: $p \in G$

By dragging the point p all over G, students can observe that for any value of r, small or large, open ball intersects G as shown in the figure below. Infact, p itself is the common point. Hence every point of G is a closure point of G.

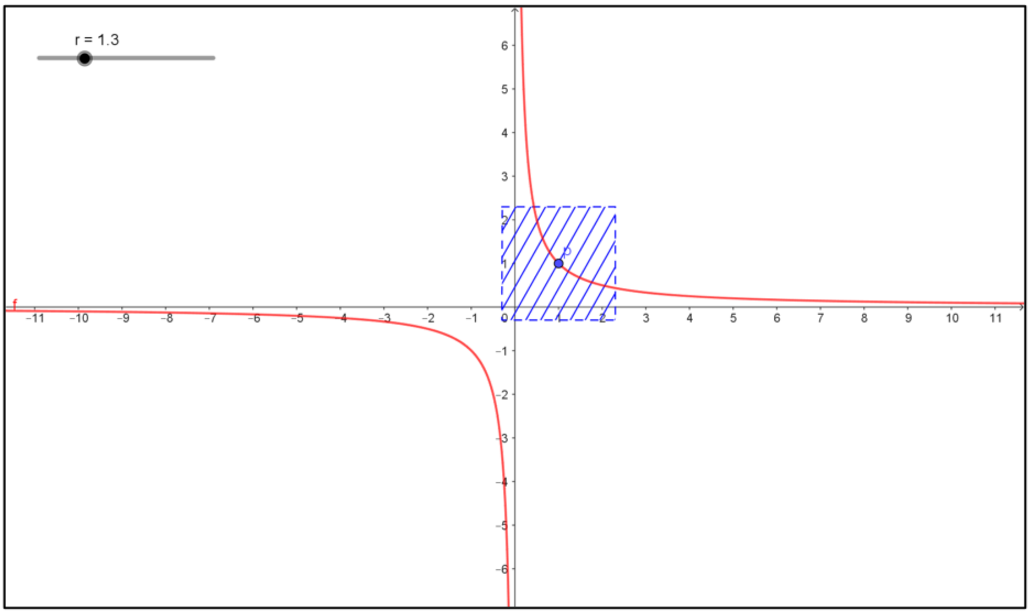

Figure 7: Case 1

This example can be further used by teacher to demonstrate the proof of the property $A \subseteq \bar{A}$. That is, every point of A is a closure point of A.

Case 2: $p \notin G$

For any point not in G, students can reduce the radius using slider till we get an open ball which does not intersect with G. They will observe that any $p \notin G$ is not a closure point of G as shown in the figure below.

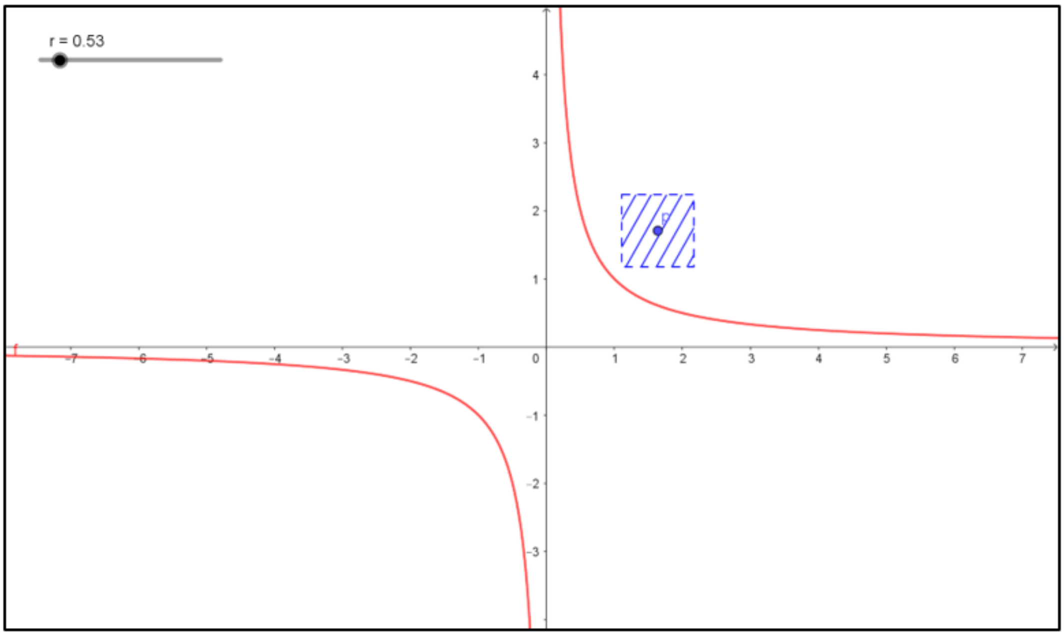

Figure 8: Case 2

Teachers should further encourage students to run the applet for different points $p \in G$ and note the radius for which the open ball does not intersect G and try to obtain a relation between p and r. Thus it will give the value of r in terms of p and help students write the proof for this case. Hence observing both the cases, students can conclude that the closure set of G is G itself.

4. CONCLUSION

GeoGebra's dynamic visualization capabilities can help teachers create engaging and interactive learning experiences that enhance students' comprehension of closure points and related concepts in metric spaces. Additionally, GeoGebra's open-source nature allows for customization and adaptation to suit the specific needs of different disciplines and teaching styles. Teachers can use the link provided in this paper for their teaching or they can prepare more examples in different metric spaces. Further, GeoGebra can help visualize proofs related to closure points in metric spaces. Teachers can animate step-by-step constructions or illustrate counterexamples to help students understand key concepts and theorems. Interactive proofs can enhance engagement and facilitate deeper understanding of the material. Interactive exploration and visualization using GeoGebra can foster a deeper understanding of abstract mathematical ideas of metric spaces.

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